Name of College – S.S. College, J.Bad

Deptt – Mathematics

Topic – Alternating Series Test
(Real Analysis)

Time —

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BY – Dr. Shree Nath Sharma
(Associate Prof)

Geometric Series :→ Important fact about Geometric Series :→

Let Geometric Series

$$1+x+x^2+x^3+\cdots+x^n+\cdots$$

If $x>1$ the Geometric Series is divergent and diverges to ∞

If $x=1$ the Geometric Series is divergent and diverges to ∞

If $-1<x<1$
the Given Series is convergent

If $x=-1$ the Given series is Finitely oscillating

If $x<-1$
the Given series is Infinitely oscillating

Here x = common ratio of Geometric Series.

Leibnitz Test (Alternating Series Test)

Statement :→ An infinite Series in Which the Terms are alternately Positive and Negative is Convergent if each Term is

Numerically less than preceding term and

$$\lim_{n\to\infty} u_n = 0$$

or

the infinite series $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ is convergent if

$u_n > 0$, $u_n < u_{n-1}$

and $\lim_{n\to\infty} u_n = 0$

<u>Proof</u> :→ Let the series be denoted by

$$u_1 - u_2 + u_3 - u_4 + \cdots$$

Where $u_1 > u_2 > u_3 > u_4 \cdots$

<u>Now</u> the given series can be rewritten in the following two ways

$$(u_1 - u_2) + (u_3 - u_4) + (u_5 - u_6) + \cdots \quad (i)$$

and $u_1 - [(u_2 - u_3) + (u_4 - u_5) + (u_5 - u_6) + \cdots] \quad (ii)$

As $u_{n-1} > u_n$ for all values of n

So from (i), we find that the expression in each bracket is +ve and therefore sum of any number of Terms of the Given Series is +ve.

From (ii) we find that the Expression in each bracket being +ve, the Sum of any No of Terms of the Given Series is always Less than U_1, a finite quantity

Also $S_{2n+1} = S_{2n} + u_{2n+1} \; \forall n \in N$ — (iii)

Also we are Given that

$$\lim_{n\to\infty} u_n = 0$$

In particular $\lim_{n\to\infty} u_{2n+1} = 0$

∴ From (iii) we get

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n}$$

Therefore, Sum of even no of Terms = Sum of odd no of Terms

Hence the Series is not oscillating Series therefore the Given Series is Convergent

Solved Example on Leibnitz Test

Problem 1 - Test the Convergency of the Series

$$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$$

Solution :→ The Given Series is

$$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$$

In this Series we Find that

(i) Terms are alternately positive and Negative

(ii) Terms are ~~Continuous~~ Continually decreasing

As $u_n < u_{n-1}$

and (iii) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

Thus all the Conditions of alternating Series Test are Satisfied and therefore the Given Series is Convergent.

Problem 2 :→ Test the Series

$$\frac{2}{1^2} - \frac{3}{2^2} + \frac{4}{3^2} - \frac{5}{4^2} \cdots$$

Here Given Series is

$$\frac{2}{1^2} - \frac{3}{2^2} + \frac{4}{3^2} - \frac{5}{4^2} + \cdots$$

In the Given Series we find that

(i) The Terms are alternately +ve and -ve

(ii) The Terms are continually decreasing

and (iii) $u_n = \frac{n+1}{n^2}$

$$\therefore \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n+1}{n^2}$$

$$= \lim_{n\to\infty} \frac{n(1+1/n)}{n^2}$$

$$= \lim_{n\to\infty} \frac{1}{n}(1+1/n)$$

$$= 0(1+0) = 0$$

Hence all three conditions of the alt series Test are satisfied and therefore the series is convergent.

Problem 3.

Test the series

$$\sum_{n=1}^{\infty} \left[\frac{(-1)^n}{2n+1}\right]$$

Solution :→ Given series is

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$$

putting $n = 1, 2, 3 \cdots$

e have

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots\ldots$$

for this series we find

(i) The Terms are alternately Positive and Negative

(ii) The Terms are continually decreasing

$$|u_n| = \frac{1}{2n-1}$$

$$\therefore \lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \frac{1}{2n-1}$$

$$= \lim_{n\to\infty} \frac{1}{n\left[2 - 1/n\right]}$$

$$= 0 \cdot \frac{1}{2-0} = 0$$

Thus all the Three conditions of the alternating Series Test are satisfied hence the series are convergent.

Problem 4. Test the Convergency of the Series

$$1 - 2x + 3x^2 - 4x^3 + \ldots\ldots \quad \text{where } x < 1$$

Solution :→ In the Given Series we have.

(i) The Terms are alternately +ve and -ve

(ii) The Terms are continually decreasing as $x < 1$

(iii) $u_n = nx^{n-1}$

$$\therefore \lim_{n\to\infty} u_n = \lim_{n\to\infty} nx^{n-1} = \frac{1}{x} \lim_{n\to\infty} nx^n = 0 \quad \text{as } \lim_{n\to\infty} nx^n = 0$$

Hence all the conditions of alternating Series are satisfi... The series is convergent.